

Chiral fermions in noncommutative electrodynamics: renormalizability and dispersion

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Abstract

We analyze quantization of noncommutative chiral electrodynamics in the enveloping algebra formalism in linear order in noncommutativity parameter θ . Calculations show that divergences exist and cannot be removed by ordinary renormalization, however they can be removed by the Seiberg-Witten redefinition of fields. Performing redefinitions explicitly, we obtain renormalizable lagrangian and discuss the influence of noncommutativity on field propagation. Noncommutativity affects the propagation of chiral fermions only: half of the fermionic modes become massive and birefringent.

1 Introduction

The original motivation to introduce noncommutativity in the forties [1] was regularization of divergences in quantum field theory; elimination of singularities in classical field theories, in particular in gravity adjoined as a motive shortly. Till the present day however the program of renormalization through noncommutativity has not been fully carried out. Initial enthusiasm, when the subject was reopened in the nineties, decreased after negative results on renormalizability in the models defined by replacement of the ordinary product by the Moyal product [2, 3, 4]. Research afterwards diversified in many directions: various modifications of field actions, different representations of symmetry, new background manifolds were analyzed. The present status is that we understand properties of gauge and scalar fields in many details while there is still no general agreement on how noncommutative gravity should be described. There are in addition affirmative results on renormalizability of some particular models.

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In their usual versions noncommutative field theories violate Lorentz invariance, the corresponding effects are related to the magnitude of noncommutativity θ . Since there are currently many experimental searches for relativity violations, both in laboratory experiments and in astrophysical measurements, a straightforward task is to use the known data to estimate the value of noncommutativity parameters and to probe various models. As indeed, a priori it is not clear that noncommutative field theories can provide with correct models which would explain some of the observed phenomena.

A possible mechanism to describe the Lorentz violation is a modification of dispersion relations at high energies. Of particular interest is the dispersion of photons, as there is a lot of data which can be used to test it. Modifications due to noncommutativity were discussed in the literature before [5, 6, 7, 8, 9, 10], both at the classical level and accounting the leading quantum corrections. The novel feature in this paper is that we discuss it within a renormalizable model; moreover, the modification is a consequence of the requirement of renormalizability. Namely, quantizing noncommutative chiral electrodynamics we obtain that all n -point functions, including the propagators, get divergent contributions. But, compared to the usual procedure our model allows an additional possibility to yield renormalizability: the Seiberg-Witten (SW) redefinition of fields [11], which in principle changes the form of the lagrangian including the kinetic terms. In fact renormalizability of other well established models like the Grosse-Wulkenhaar model [12] was achieved in a similar manner, by changing the propagator. The result which we obtain is that the additional kinetic term in the gauge field action has no effect and the photons propagate as usual. But chiral fermions are sensitive to noncommutativity: half of the spinor modes acquire mass which is of order $1/\sqrt{\theta}$ and depends on the direction: there is vacuum birefringence. This behavior is new and very interesting.

Along with propagators, interaction vertices change, too. New processes induced by noncommutative interactions can also be used to estimate values of the parameters in the theory. We will not discuss them in this paper, leaving this issue for the future work.

The framework which we use is the ‘enveloping algebra formalism’ or the ‘ θ -expanded gauge theory’. θ stands for the value of the position commutator $\theta^{\mu\nu}$,

$$[x^\mu \star, x^\nu] \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad (1.1)$$

this is the relation which defines the flat noncommutative space. Commutator in (1.1) is the \star -commutator given in terms of the Moyal product of functions,

$$\phi(x) \star \chi(x) = e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} \phi(x)\chi(y)|_{y \rightarrow x}. \quad (1.2)$$

The $\theta^{\mu\nu}$ is a constant dimensionful tensor. It gives the length scale on which the quantum structure of spacetime becomes important; its value can be of order of the square of Planck length or larger. Having in mind smallness of θ , it makes sense to search for the effects in the leading, linear order in θ . This was in part a motivation to introduce θ -expansion in the noncommutative gauge theories originally [13]. One should keep in mind however that the issue of convergence of this expansion and relations to other theories are still open.

θ -expanded theories have interesting properties. They are based on the enlargement of the initial gauge algebra to its enveloping algebra. This permits possibility to introduce direct products of gauge groups and different charges for different particles. Further, it is known that photon self-energy is renormalizable to all orders in θ using the SW freedom in quadratic and higher orders, [14]. Moreover, pure $SU(N)$ gauge theories are perturbatively renormalizable in

θ -linear order without SW redefinition, [15, 16]. Similar holds for the gauge sector of a suitably defined generalization of the Standard Model, [17]. Still for some time it was believed that fermions cannot be successfully incorporated into a renormalizable theory because of the so-called $4\text{-}\psi$ divergence, [18]. Our main motivation to investigate chiral electrodynamics in more details is the result that $4\text{-}\psi$ divergence is only related to theories with Dirac fermions, while it vanishes for U(1) and SU(2) theory with chiral fermions, [19]. This result was generalized to arbitrary GUT-inspired models with chiral fermions in [20], and it opened a possibility to construct renormalizable gauge theories with matter. First such models were proposed in [21, 22], where their on-shell one-loop renormalizability was shown.

In this paper we continue along the same line of investigation by analyzing off-shell one-loop renormalizability of noncommutative chiral electrodynamics in linear order in θ . The results of the calculation show that divergences exist and that they cannot be removed by ordinary renormalization of coupling constants. However, divergences are of the type which can be removed by the Seiberg-Witten redefinition of fields. We perform this redefinition explicitly and analyze the modification of the propagation properties of fields in our model.

2 Noncommutative chiral electrodynamics

The commutative action for chiral electrodynamics is given by

$$S_C = \int d^4x \left(i\bar{\varphi}\bar{\sigma}^\mu(\partial_\mu + iqA_\mu)\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right), \quad (2.1)$$

where φ denotes the left chiral fermion, q is its charge, A_μ is the U(1) vector potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the corresponding field strength. The noncommutative U(1) symmetry can be realized by an analogous set of fields which we denote by a hat: $\hat{\varphi}$, \hat{A}_μ and $\hat{F}_{\mu\nu}$. As noncommutative U(1) group is nonabelian, the field strength is $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + iq[\hat{A}_\mu, \hat{A}_\nu]$; otherwise all definitions in the two cases are analogous.

Commutative and noncommutative symmetries which correspond to the same gauge group can be related by the Seiberg-Witten map: the map gives an explicit relation between corresponding gauge and matter fields. SW map can also be seen as an expansion in $\theta^{\mu\nu}$

$$\hat{A}_\mu = \sum A_\mu^{(n)}, \quad \hat{\varphi} = \sum \varphi^{(n)}, \quad (2.2)$$

where terms $A_\mu^{(n)}$ and $\varphi^{(n)}$ contain $\theta^{\mu\nu}$ to the n -th power. One also assumes that $A_\mu^{(0)} = A_\mu$, $\varphi^{(0)} = \varphi$, which fixes the commutative limit $\theta^{\mu\nu} = 0$ of the theory.

Seiberg-Witten expansion (2.2) can be seen as a solution to the group closure equations. The simplest solution to linear order is [13, 23]:

$$\hat{A}_\rho = A_\rho + \frac{1}{4}q\theta^{\mu\nu}\{A_\mu, \partial_\nu A_\rho + F_{\nu\rho}\}, \quad (2.3)$$

$$\hat{F}_{\rho\sigma} = F_{\rho\sigma} - \frac{1}{2}q\theta^{\mu\nu}\{F_{\mu\rho}, F_{\nu\sigma}\} + \frac{1}{4}q\theta^{\mu\nu}\{A_\mu, (\partial_\nu + D_\nu)F_{\rho\sigma}\}, \quad (2.4)$$

$$\hat{\varphi} = \varphi + \frac{1}{2}q\theta^{\mu\nu}A_\mu\partial_\nu\varphi, \quad (2.5)$$

where D_μ denotes the commutative covariant derivative, $D_\mu\varphi = (\partial_\mu + iqA_\mu)\varphi$. However, this solution is not unique. It was shown in [24, 14] that a whole class of solutions can be

obtained from (2.3-2.5) by a shift of fields

$$A_\mu^{(n)} \rightarrow A_\mu^{(n)} + \mathbf{A}_\mu^{(n)}, \quad \varphi^{(n)} \rightarrow \varphi^{(n)} + \mathbf{\Phi}^{(n)}, \quad (2.6)$$

where $\mathbf{A}_\mu^{(n)}$ and $\mathbf{\Phi}^{(n)}$ are arbitrary gauge covariant expressions of given order n , $n > 0$. This means that, if we assume that noncommutative fields \hat{A}_μ and $\hat{\varphi}$ are primary or ‘physical’ objects in the theory and likewise, that noncommutative action is fixed by a first principle, when written in commutative fields A_μ , φ the action is not unique. One needs an additional criterion to decide which of the induced commutative actions is physical. At the same time, nonuniqueness gives a new family of counterterms which can be used to achieve renormalizability of the theory.

The action for noncommutative chiral electrodynamics is given by

$$S_{\text{NC}} = \int d^4x \left(i\hat{\varphi} \star \bar{\sigma}^\mu (\partial_\mu + iq\hat{A}_\mu) \star \hat{\varphi} - \frac{1}{4} \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} \right). \quad (2.7)$$

This action can be expanded in commutative fields and then quantized by the usual methods. We truncate the expansion at linear order in θ . Using (2.3-2.5) we obtain

$$\mathcal{L}_{\text{NC}} = \mathcal{L}_0 + \mathcal{L}_{1,A} + \mathcal{L}_{1,\varphi}, \quad (2.8)$$

with

$$\mathcal{L}_0 \equiv \mathcal{L}_C = i\bar{\varphi} \bar{\sigma}^\mu (D_\mu \varphi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.9)$$

$$\mathcal{L}_{1,A} = \frac{1}{2} q \theta^{\mu\nu} \left(F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \quad (2.10)$$

$$\mathcal{L}_{1,\varphi} = \frac{i}{4} q \left(\theta^{\mu\nu} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) - 2\theta^{\mu\nu} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi) \right) \quad (2.11)$$

$$= \frac{i}{16} q \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} F_{\alpha\beta} \bar{\varphi} \bar{\sigma}^\rho (D_\gamma \varphi) + \text{h.c.} \quad (2.12)$$

Cyclic and antisymmetric Δ in (2.12) is defined by $\Delta_{\mu\nu\rho}^{\alpha\beta\gamma} = -\varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\delta}$. Though (2.8) is not unique we start with it because it is the simplest of the actions. In principle it is possible to treat the whole class of actions from the beginning (e.g. it was done for a similar model in [25]), but such approach introduces a large number of coupling constants and makes an already difficult calculation very complicated.

3 Quantization

We quantize action (2.7) by using the path integral method. Concrete details of the method and of our notation can be found in [26, 27, 19], we will stress here only some specific points. In principle, θ -dependent terms are treated as interactions and $\theta^{\mu\nu}$ as a coupling constant. Since the interaction terms in (2.8) contain three and more fields, propagators for the spinor and for the gauge fields are the same as in commutative theory. To compute the functional integral one has to complexify the gauge potential or to introduce the Majorana spinors instead of the chiral; we do the latter. Denoting the Majorana spinor by ψ ,

$$\psi = \begin{pmatrix} \varphi_\alpha \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix}, \quad (3.1)$$

we can write the commutative part of the lagrangian in the form

$$\mathcal{L}_0 = \frac{i}{2} \bar{\psi} \gamma^\mu (\partial_\mu - iq\gamma_5 A_\mu) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (3.2)$$

Expressions (3.2) and (2.9) are identical in the chiral representation of γ -matrices (A.2). The θ -linear spinor part of the lagrangian is expressed as

$$\mathcal{L}_{1,\varphi} = \frac{i}{16} q \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} F_{\alpha\beta} \bar{\psi} \gamma^\rho (\partial_\gamma - iq\gamma_5 A_\gamma) \psi, \quad (3.3)$$

while the gauge part is of course the same, (2.10).

In order to preserve the gauge covariance we use the background field method. Briefly: we expand fields around their classical configurations, replacing formally in the action $A_\mu \rightarrow A_\mu + \mathcal{A}_\mu$ and $\psi \rightarrow \psi + \Psi$; then we integrate over the quantum fields \mathcal{A}_μ, Ψ . After the integration we obtain the one-loop effective action Γ ,

$$\Gamma[A_\mu, \psi] = S_{cl}[A_\mu, \psi] - \frac{1}{2i} \text{STr} \log \mathcal{B}[A_\mu, \psi]. \quad (3.4)$$

The first term is the classical part, the second term is the one-loop quantum correction. Operator \mathcal{B} , the result of Gaussian integration, can be obtained as the term of second order in the expansion of classical action $S_{cl}[A_\mu + \mathcal{A}_\mu, \psi + \Psi]$ in fields \mathcal{A}_μ and Ψ ,

$$S^{(2)} = \int d^4x \begin{pmatrix} \mathcal{A}_\kappa & \bar{\Psi} \end{pmatrix} \mathcal{B} \begin{pmatrix} \mathcal{A}_\lambda \\ \Psi \end{pmatrix}. \quad (3.5)$$

\mathcal{B} can be divided into commutative part \mathcal{B}_0 and θ -linear part \mathcal{B}_1 , $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$. After inclusion of the gauge fixing terms, \mathcal{B}_0 is given by

$$\mathcal{B}_0 = \frac{1}{2} \begin{pmatrix} g^{\kappa\lambda} \square & q \bar{\psi} \gamma^\kappa \gamma_5 \\ q \gamma^\lambda \gamma_5 \psi & i \not{\partial} + q \not{A} \gamma_5 \end{pmatrix}; \quad (3.6)$$

it has the kinetic part $\mathcal{B}_{\text{kin}} = \frac{1}{2} \begin{pmatrix} g^{\kappa\lambda} \square & 0 \\ 0 & i \not{\partial} \end{pmatrix}$ and the interaction.

In order to calculate the one-loop effective action perturbatively we need to expand the logarithm in (3.4) around identity $\mathcal{I} = \begin{pmatrix} g^{\kappa\lambda} & 0 \\ 0 & 1 \end{pmatrix}$. Thus we have to multiply \mathcal{B} by matrix \mathcal{C} ,

$$\mathcal{C} = 2 \begin{pmatrix} g^{\kappa\lambda} & 0 \\ 0 & -i \not{\partial} \end{pmatrix}; \quad (3.7)$$

we then have $\mathcal{B}_{\text{kin}} \mathcal{C} = \mathcal{I}$ and

$$\text{STr} (\log \mathcal{B}) = \text{STr} (\log \square^{-1} \mathcal{B} \mathcal{C}) - \text{STr} (\mathcal{C} \square^{-1}). \quad (3.8)$$

The first term in (3.8) we denote by $\Gamma^{(1)}$ as, clearly, up to a constant infinite normalization $\text{STr} (\mathcal{C} \square^{-1})$, it can be identified with first quantum correction of the one-loop effective action. Introducing

$$\mathcal{B} \mathcal{C} = \square \mathcal{I} + N_1 + T_1 + T_2, \quad (3.9)$$

we obtain the perturbation expansion

$$\begin{aligned}\Gamma^{(1)} &= \frac{i}{2} \text{STr} \log (\mathcal{I} + \square^{-1} N_1 + \square^{-1} T_1 + \square^{-1} T_2) \\ &= \frac{i}{2} \sum \frac{(-1)^{n+1}}{n} \text{STr} (\square^{-1} N_1 + \square^{-1} T_1 + \square^{-1} T_2)^n.\end{aligned}\quad (3.10)$$

Interaction in (3.9) is divided in three parts in the following way. Operator N_1 contains commutative vertices; in case of electrodynamics there is only a 3-vertex, so N_1 contains terms with one classical (external) and two quantum fields. By analogy, T_1 is a term linear in θ which contains one classical field and two quantum fields. T_2 is linear in θ and contains two classical and two quantum fields. From (3.6) we obtain

$$N_1 = q \begin{pmatrix} 0 & -i\bar{\psi}\gamma_5\gamma^\lambda\partial \\ -\gamma_5\gamma^\kappa\psi & i\gamma_5 A\partial \end{pmatrix}, \quad (3.11)$$

$$T_1 = -q \begin{pmatrix} V^{\kappa\lambda} & -\frac{1}{4}\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\delta_\alpha^\kappa(\partial_\beta\bar{\psi})\gamma^\rho\partial_\gamma\partial \\ -\frac{i}{4}\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\delta_\alpha^\lambda\gamma^\rho(\partial_\beta\psi)\partial_\gamma & -\frac{1}{8}\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}F_{\alpha\beta}\gamma^\rho\partial_\gamma\partial \end{pmatrix}, \quad (3.12)$$

where the matrix elements $V^{\kappa\lambda}$ are given as $V^{\kappa\lambda} = -\partial_\sigma V^{\sigma\kappa,\tau\lambda}\partial_\tau$ with, [26],

$$\begin{aligned}V^{\sigma\kappa,\tau\lambda} &= \frac{1}{2}(g^{\sigma\tau}g^{\kappa\lambda} - g^{\sigma\lambda}g^{\tau\kappa})\theta^{\alpha\beta}F_{\alpha\beta} \\ &\quad - g^{\kappa\lambda}(\theta^{\xi\sigma}F_\xi^\tau + \theta^{\xi\tau}F_\xi^\sigma) - g^{\sigma\tau}(\theta^{\xi\kappa}F_\xi^\lambda + \theta^{\xi\lambda}F_\xi^\kappa) \\ &\quad + g^{\kappa\tau}(\theta^{\xi\lambda}F_\xi^\sigma + \theta^{\xi\sigma}F_\xi^\lambda) + g^{\sigma\lambda}(\theta^{\xi\kappa}F_\xi^\tau + \theta^{\xi\tau}F_\xi^\kappa) \\ &\quad - \theta^{\kappa\lambda}F^{\sigma\tau} + \theta^{\kappa\tau}F^{\sigma\lambda} + \theta^{\sigma\lambda}F^{\kappa\tau} + \theta^{\sigma\kappa}F^{\tau\lambda} + \theta^{\tau\lambda}F^{\sigma\kappa} - \theta^{\sigma\tau}F^{\kappa\lambda}.\end{aligned}\quad (3.13)$$

T_2 is equal to

$$T_2 = \frac{1}{8}q^2\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma} \begin{pmatrix} \delta_\alpha^\kappa\delta_\beta^\lambda(\partial_\gamma\bar{\psi}\gamma_5\gamma^\rho\psi + \bar{\psi}\gamma_5\gamma^\rho\psi\partial_\gamma) & i\delta_\alpha^\kappa(2\partial_\beta A_\gamma + F_{\beta\gamma})\bar{\psi}\gamma_5\gamma^\rho\partial \\ \delta_\alpha^\lambda\gamma_5\gamma^\rho\psi(2A_\gamma\partial_\beta - F_{\beta\gamma}) & iF_{\alpha\beta}A_\gamma\gamma_5\gamma^\rho\partial \end{pmatrix}. \quad (3.14)$$

The order of operators is of importance; in our notation we have for example

$$\begin{aligned}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\partial_\beta A_\gamma &= \Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\left((\partial_\beta A_\gamma) + A_\gamma\partial_\beta\right) = \Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\left(\frac{1}{2}F_{\beta\gamma} + A_\gamma\partial_\beta\right), \\ \partial_\gamma\bar{\psi}\gamma_\rho\gamma_5\psi &= (\partial_\gamma(\bar{\psi}\gamma_\rho\gamma_5\psi)) + \bar{\psi}\gamma_\rho\gamma_5\psi\partial_\gamma\end{aligned}\quad (3.15)$$

and so on.

4 Divergences and renormalization

We would like to extract divergent parts of the one-loop effective action from expansion (3.10); the relevant terms can be identified by power counting. Divergences exist in the 2-point, 3-point and 4-point functions and a careful analysis shows that they are contained only in terms $\text{STr}(\square^{-1}N_1\square^{-1}T_1)$, $\text{STr}(\square^{-1}N_1\square^{-1}T_2)$ and $\text{STr}(\square^{-1}N_1\square^{-1}N_1\square^{-1}T_1)$. Supertraces

can be calculated in a standard fashion, in the momentum representation using dimensional regularization. The calculation itself however is very demanding and to obtain the results we combined ordinary calculation with the algebraic one using the *MathTensor* package in *Mathematica*. To separate contributions coming from different n -point functions we denote

$$\Gamma^{(1)}|_{\text{div}} = \Gamma_2 + \Gamma_3 + \Gamma_4 = \Gamma_2 + \Gamma_3. \quad (4.1)$$

Γ_2 and Γ_3 will be used later and they denote divergences written in a covariant form. The divergent part of the 2-point function is:

$$\begin{aligned} \Gamma_2 &= -\frac{i}{2} \text{STr}(\square^{-1} N_1 \square^{-1} T_1)|_{\text{div}} \\ &= -\frac{i}{8} q^2 \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} \text{Tr}(\square^{-1} \bar{\psi} \gamma_5 \gamma_\alpha \not{\partial} \square^{-1} \gamma^\rho (\partial_\gamma \psi) \partial_\beta + \square^{-1} \gamma_5 \gamma_\alpha \psi \square^{-1} (\partial_\beta \bar{\psi}) \gamma^\rho \partial_\gamma \not{\partial}) \\ &\quad - \frac{1}{16} q^2 \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} \text{Tr}(\square^{-1} \gamma_5 A \not{\partial} \square^{-1} F_{\alpha\beta} \gamma^\rho \partial_\gamma \not{\partial}) \\ &= -\frac{i}{48} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} (\partial_\gamma \bar{\psi}) (\gamma_5 \gamma_\alpha \gamma_\beta \gamma^\rho - \gamma_5 \gamma^\rho \gamma_\beta \gamma_\alpha) (\square \psi) \\ &\quad - \frac{i}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} \varepsilon^{\sigma\kappa\lambda\rho} A_\kappa \eta_{\lambda\gamma} (\partial_\sigma \square F_{\alpha\beta}), \end{aligned} \quad (4.2)$$

so we obtain

$$\Gamma_2 = -\frac{1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(i \varepsilon_{\mu\nu}{}^{\rho\sigma} (\partial_\rho \bar{\psi}) \gamma_\sigma (\square \psi) + \varepsilon_\mu{}^{\rho\sigma\tau} F_{\rho\sigma} (\square F_{\nu\tau}) \right). \quad (4.3)$$

Calculation of the divergent parts of the 3-point functions gives:

$$\begin{aligned} \text{STr}(\square^{-1} N_1 \square^{-1} T_2)|_{\text{div}} &= -\frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(-\frac{i}{3} A_\rho (\partial_\mu \bar{\psi}) \gamma^\rho (\partial_\nu \psi) - \frac{i}{3} A_\mu (\partial_\nu \bar{\psi}) \gamma^\rho (\partial_\rho \psi) \right. \\ &\quad - \frac{i}{3} A^\rho (\partial_\mu \bar{\psi}) \gamma_\nu (\partial_\rho \psi) - \frac{1}{3} \varepsilon_\mu{}^{\rho\sigma\tau} A_\rho (\partial_\nu \bar{\psi}) \gamma_5 \gamma_\sigma (\partial_\tau \psi) - \frac{4i}{3} F_{\mu\rho} \bar{\psi} \gamma^\rho (\partial_\nu \psi) \\ &\quad - \frac{4i}{3} F_{\mu\rho} \bar{\psi} \gamma_\nu (\partial^\rho \psi) - \frac{4}{3} \varepsilon_\mu{}^{\rho\sigma\tau} F_{\nu\rho} \bar{\psi} \gamma_5 \gamma_\sigma (\partial_\tau \psi) - \frac{2i}{3} A_\mu \bar{\psi} \gamma_\nu (\square \psi) \\ &\quad \left. + \frac{1}{6} \varepsilon_{\mu\nu}{}^{\rho\sigma} A_\rho \bar{\psi} \gamma_5 \gamma_\sigma (\square \psi) + \frac{2}{3} A^\rho F_{\mu\nu} (\partial_\sigma F_{\rho\sigma}) - \frac{4}{3} A_\mu F_{\nu\rho} (\partial_\sigma F^{\rho\sigma}) \right), \\ \text{STr}(\square^{-1} N_1 \square^{-1} N_1 \square^{-1} T_1)|_{\text{div}} &= -\frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(-\frac{i}{3} A_\rho (\partial_\mu \bar{\psi}) \gamma^\rho (\partial_\nu \psi) - \frac{i}{3} A_\mu (\partial_\nu \bar{\psi}) \gamma^\rho (\partial_\rho \psi) \right. \\ &\quad - \frac{5i}{3} A^\rho (\partial_\mu \bar{\psi}) \gamma_\nu (\partial_\rho \psi) - \frac{1}{3} \varepsilon_\mu{}^{\rho\sigma\tau} A_\rho (\partial_\nu \bar{\psi}) \gamma_5 \gamma_\sigma (\partial_\tau \psi) + \frac{i}{3} F_{\mu\rho} \bar{\psi} \gamma^\rho (\partial_\nu \psi) \\ &\quad + \frac{2i}{3} F_{\mu\rho} \bar{\psi} \gamma_\nu (\partial^\rho \psi) + \frac{i}{6} F_{\mu\nu} \bar{\psi} \gamma^\rho (\partial_\rho \psi) - \frac{2i}{3} (\partial_\rho A^\rho) \bar{\psi} \gamma_\mu (\partial_\nu \psi) \\ &\quad + \frac{1}{3} \varepsilon_{\mu\nu}{}^{\rho\sigma} A^\tau (\partial_\rho \bar{\psi}) \gamma_5 \gamma_\sigma (\partial_\tau \psi) - \frac{1}{6} \varepsilon_{\mu\nu}{}^{\rho\sigma} (\partial_\tau A^\tau) \bar{\psi} \gamma_5 \gamma_\rho (\partial_\sigma \psi) + \frac{1}{12} \varepsilon_\mu{}^{\rho\sigma\tau} F_{\rho\sigma} \bar{\psi} \gamma_5 \gamma_\tau (\partial_\nu \psi) \\ &\quad \left. + \frac{1}{6} \varepsilon_\mu{}^{\rho\sigma\tau} F_{\nu\rho} \bar{\psi} \gamma_5 \gamma_\sigma (\partial_\tau \psi) - \frac{1}{4} \varepsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau (\partial_\tau \psi) \right), \end{aligned}$$

so the result for the divergent 3-vertices is given by:

$$\begin{aligned}
\Gamma_3 = & \frac{i}{2} \left(\text{STr}(\square^{-1} N_1 \square^{-1} N_1 \square^{-1} T_1) \Big|_{\text{div}} - \text{STr}(\square^{-1} N_1 \square^{-1} T_2) \Big|_{\text{div}} \right) \\
= & -\frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(\frac{1}{6} F_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} - \frac{2}{3} F_{\mu\rho} F^{\nu\sigma} F_{\rho\sigma} + \frac{5i}{6} F_{\mu\rho} \bar{\psi} \gamma^\rho (\partial_\nu \psi) \right. \\
& - \frac{i}{6} F_{\mu\rho} \bar{\psi} \gamma_\nu (\partial^\rho \psi) - \frac{2i}{3} F_{\mu\nu} \bar{\psi} \gamma^\rho (\partial_\rho \psi) + \frac{4}{3} \varepsilon_\mu^{\rho\sigma\tau} F_{\rho\sigma} \bar{\psi} \gamma_5 \gamma_\tau (\partial_\nu \psi) \\
& + \frac{3}{2} \varepsilon_{\mu\nu}^{\rho\sigma} F_{\rho\tau} \bar{\psi} \gamma_5 \gamma_\sigma (\partial^\tau \psi) + \frac{1}{8} \varepsilon_{\mu\nu}^{\rho\sigma} F_{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau (\partial_\tau \psi) + \frac{1}{12} \varepsilon_{\mu\nu}^{\rho\sigma} A_\rho \bar{\psi} \gamma_5 \gamma_\sigma (\square \psi) \\
& \left. - \frac{1}{6} \varepsilon_{\mu\nu}^{\rho\sigma} A^\tau (\partial_\rho \bar{\psi}) \gamma_5 \gamma_\sigma (\partial_\tau \psi) + \frac{1}{12} \varepsilon_{\mu\nu}^{\rho\sigma} (\partial_\tau A^\tau) \bar{\psi} \gamma_5 \gamma_\rho (\partial_\sigma \psi) \right). \tag{4.4}
\end{aligned}$$

It would be more transparent to have these expressions written in covariant derivatives, however that is not possible in the Majorana representation. Therefore we rewrite (4.3) and (4.4) in the chiral representation, collecting together covariant pieces. This mixes Γ_2 and Γ_3 ; for example, the last three terms of (4.4) belong in fact to the 2-point function when we write it in covariant derivatives. The divergences become

$$\Gamma_2 = \frac{1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\rho\sigma\tau} (\partial_\lambda F^{\rho\lambda}) (\partial_\nu F^{\sigma\tau}) - \frac{1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} (i(D^\rho \bar{\varphi}) \bar{\sigma}^\sigma (D^2 \varphi) + \text{h.c.}), \tag{4.5}$$

and

$$\begin{aligned}
\Gamma_3 = & -\frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(\frac{1}{6} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - \frac{2}{3} F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} \right) \\
& - \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(\frac{5i}{6} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi) - \frac{i}{6} F_{\mu\rho} \bar{\varphi} \bar{\sigma}_\nu (D^\rho \varphi) - \frac{2i}{3} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) \right. \\
& + \frac{4}{3} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau (D_\nu \varphi) + \frac{3}{2} \varepsilon_{\mu\nu\rho\tau} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau (D_\sigma \varphi) \\
& \left. + \frac{1}{8} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau (D_\tau \varphi) + \text{h.c.} \right). \tag{4.6}
\end{aligned}$$

The last formula contains all vertex terms, including the 4-point functions Γ_4 . From the form of (4.5) and (4.6) it is quite clear that divergences cannot be removed by multiplicative renormalization: there are for example divergent contributions to the propagators while both propagating fields are massless. Also interaction terms in Γ_3 have not the form of the interaction lagrangian $\mathcal{L}_{1,\varphi}$. Therefore the only possibility for renormalization is the Seiberg-Witten redefinition of fields, and we will explore this possibility more closely. Of course the analysis which follows is, as all other calculations, done only in linear order in θ . A general SW redefinition (2.6) induces in the action the following additional terms:

$$\Delta S^{(n,A)} = \int d^4x (D_\rho F^{\rho\mu}) \mathbf{A}_\mu^{(n)}, \tag{4.7}$$

$$\Delta S^{(n,\varphi)} = i \int d^4x \bar{\varphi} \bar{\sigma}^\mu (D_\mu \Phi^{(n)}) + \text{h.c.}, \tag{4.8}$$

so we need to rewrite Γ_2 and Γ_3 in such form, of course except for the terms proportional to $\mathcal{L}_{1,A}$ and $\mathcal{L}_{1,\varphi}$. We can see immediately that the bosonic part of the two-point divergence is already in the required form. The shift of the gauge potential $A_\rho \rightarrow A_\rho + \mathbf{A}_\rho$, with

$$\mathbf{A}_\rho = \frac{1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\rho\sigma\tau} (D_\nu F^{\sigma\tau}) \quad (4.9)$$

cancels it. In fact a similar shift of the spinor φ , $\varphi \rightarrow \varphi + \Phi$, can be done to cancel the fermionic part of Γ_2 , too. Using relations (A.4) among σ and $\bar{\sigma}$ -matrices we obtain that for

$$\Phi = -\frac{i}{6} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \sigma_{\mu\nu} (D^2 \varphi) \quad (4.10)$$

the effective action transforms to

$$\Gamma = S_{\text{cl}} + \Gamma_2 + \Gamma_3 \rightarrow S_{\text{cl}} + \Gamma_3 + \Gamma'_3, \quad (4.11)$$

removing Γ_2 completely. Redefinitions (4.9)-(4.10) along with the cancellation of Γ_2 induce an additional term Γ'_3 ,

$$\Gamma'_3 = -\frac{1}{12} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(2i F_{\nu\rho} \bar{\varphi} \bar{\sigma}_\mu (D_\rho \varphi) - \varepsilon_{\mu\rho\sigma\tau} F^{\sigma\tau} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi) \right) + \text{h.c.}, \quad (4.12)$$

and therefore in the next step we have to ‘redefine away’ the 3-point divergence

$$\begin{aligned} \Gamma_3 + \Gamma'_3 &= \frac{4}{3} \frac{q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,A} \\ &\quad - \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(\frac{5i}{6} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi) - \frac{i}{3} F_{\mu\rho} \bar{\varphi} \bar{\sigma}_\nu (D^\rho \varphi) - \frac{2i}{3} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) + \text{h.c.} \right) \\ &\quad - \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau \left(\frac{5}{4} \varepsilon_{\mu\rho\sigma\tau} (D_\nu \varphi) + \frac{3}{2} \varepsilon_{\mu\nu\rho\tau} (D_\sigma \varphi) + \frac{1}{8} \varepsilon_{\mu\nu\rho\sigma} (D_\tau \varphi) + \text{h.c.} \right). \end{aligned} \quad (4.13)$$

At the first sight it looks as if there were too many terms in (4.13) to cancel: six. However, by inspecting them separately we can see that this is not the case. For example it is obvious that the last terms in the second and in the third line, $F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi)$ and $\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau (D_\tau \varphi)$, are already in the form adjusted for the spinor field redefinition; this leaves four terms. Let us first discuss the second line of expression (4.13). We observe that it is possible to combine terms to obtain $\mathcal{L}_{1,\varphi}$; for example, $\theta^{\mu\nu} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi)$ can be replaced with

$$\theta^{\mu\nu} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi) = \frac{1}{2} \theta^{\mu\nu} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) - \frac{1}{4} \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} F_{\alpha\beta} \bar{\varphi} \bar{\sigma}^\rho (D_\gamma \varphi), \quad (4.14)$$

that is we can use

$$i \theta^{\mu\nu} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho (D_\nu \varphi) + \text{h.c.} = \frac{i}{2} \theta^{\mu\nu} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) + \text{h.c.} - \frac{4}{q} \mathcal{L}_{1,\varphi}. \quad (4.15)$$

Therefore in fact there is only one nontrivial term in the second line of (4.13) and it can be removed by a spinor shift

$$\Phi' = \frac{2}{3} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} F_{\mu\rho} \sigma_{\nu\rho} \varphi. \quad (4.16)$$

This gives additional contribution Γ_3'' to Γ , so after the second shift (4.16) we have

$$\begin{aligned} \Gamma_3 + \Gamma_3' + \Gamma_3'' &= \frac{4}{3} \frac{q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,A} + \frac{2q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,\varphi} + \frac{5i}{12} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \left(F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) + \text{h.c.} \right) \\ &\quad - \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau \left(\frac{17}{12} \varepsilon_{\mu\rho\sigma\tau} (D_\nu \varphi) + \frac{3}{2} \varepsilon_{\mu\nu\rho\tau} (D_\sigma \varphi) + \frac{1}{8} \varepsilon_{\mu\nu\rho\sigma} (D_\tau \varphi) + \text{h.c.} \right). \end{aligned} \quad (4.17)$$

The last line of formula (4.17) again has three terms but only two are independent: this time due to identity (A.10). Furthermore, $\varepsilon_{\mu\nu\rho\tau} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau (D_\sigma \varphi)$ can be cancelled by a redefinition of the gauge field \mathbf{A}'_ρ ,

$$\mathbf{A}'_\rho = -\frac{1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\tau} (\partial^\sigma F_{\tau\sigma}), \quad (4.18)$$

which does not change the bosonic part of the action. Thus after transformation $A_\rho \rightarrow A_\rho + \mathbf{A}_\rho + \mathbf{A}'_\rho$, $\varphi \rightarrow \varphi + \Phi + \Phi'$ we obtain

$$\begin{aligned} \Gamma &\rightarrow S_{\text{cl}} + \Gamma_3 + \Gamma_3' + \Gamma_3'' = \\ &= S_{\text{cl}} + \frac{4}{3} \frac{q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,A} + \frac{2q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,\varphi} \\ &\quad + \frac{5i}{12} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho (D_\rho \varphi) - \frac{5}{6} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} F^{\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau (D_\tau \varphi) + \text{h.c.} . \end{aligned} \quad (4.19)$$

The last two shifts, \mathbf{A}''_ρ (which is a gauge transformation) and Φ'' ,

$$\begin{aligned} \mathbf{A}''_\rho &= -\frac{5}{6} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\nu\tau\sigma} (\partial_\rho F^{\tau\sigma}), \\ \Phi'' &= -\frac{5}{12} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} F_{\mu\nu} \varphi - \frac{5i}{6} \frac{q^3}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \varphi, \end{aligned} \quad (4.20)$$

transform the effective action to

$$\Gamma = S_{\text{cl}} + \frac{4}{3} \frac{q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,A} + \frac{2q^2}{(4\pi)^2 \epsilon} \mathcal{L}_{1,\varphi}. \quad (4.21)$$

The remaining divergence can be removed by a multiplicative renormalization of fields and coupling constants; noncommutativity parameter θ in principle gets renormalized, too.

5 Discussion

We have seen in the previous section that it is possible to find a Seiberg-Witten redefinition of commutative fields A_ρ and φ

$$A_\rho \rightarrow A_\rho + \mathbf{A}_\rho + \mathbf{A}'_\rho + \mathbf{A}''_\rho, \quad \varphi \rightarrow \varphi + \Phi + \Phi' + \Phi'', \quad (5.1)$$

which cancels all divergences in the one-loop correction to the effective action $\Gamma^{(1)}$, except for classical interaction terms. This redefinition changes neither the physical noncommutative theory nor its commutative limit; it changes only the identification of the θ -linear part of the

action in terms of commutative fields A_ρ and φ . This means that, had we taken instead of the simplest expansions (2.3-2.5), the other defined by

$$\hat{A}_\rho = A_\rho + \frac{1}{4}q\theta^{\mu\nu}\{A_\mu, \partial_\nu A_\rho + F_{\nu\rho}\} + a\mathbf{A}_\rho + a'\mathbf{A}'_\rho + a''\mathbf{A}''_\rho, \quad (5.2)$$

$$\hat{\varphi} = \varphi + \frac{1}{2}q\theta^{\mu\nu}A_\mu\partial_\nu\varphi + b\Phi + b'\Phi' + b''\Phi'', \quad (5.3)$$

we would have obtained one-loop renormalizable action of the form¹

$$\begin{aligned} \mathcal{L}_{\text{NC}} = & \mathcal{L}_{\text{C}} + \kappa_1\mathcal{L}_{1,A} + \kappa_2\mathcal{L}_{1,\varphi} \\ & + \kappa_3\theta^{\mu\nu}\varepsilon_\mu^{\rho\sigma\tau}F_{\rho\sigma}(D^2F_{\nu\tau}) + i\kappa_4\theta^{\mu\nu}\left(i\bar{\varphi}\bar{\sigma}_\rho\sigma_{\mu\nu}(D^\rho D^2\varphi) + \text{h.c.}\right) \\ & + \theta^{\mu\nu}\bar{\varphi}\left((\kappa_5F_{\mu\nu} + \kappa_6\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} + \kappa_7F_\mu{}^\rho\sigma_{\nu\rho})\varphi + \text{h.c.}\right). \end{aligned} \quad (5.4)$$

To prove the last statement rigorously one should in fact start with (5.4), repeat all steps of quantization, renormalize couplings κ_i and $\theta^{\mu\nu}$ explicitly, find β -functions, etc. This we will do in our following work. However, already calculations presented here strongly indicate renormalizability because, due to various identities, all divergent terms of appropriate dimension which could be obtained are already included in (5.4).

In comparison with (2.8), lagrangian (5.4) contains new interaction vertices: these are terms proportional to κ_5 , κ_6 and κ_7 . It contains also a modification of propagators. The change of the photon dispersion relation is perhaps more interesting because one hopes to compare its effects with the data on anisotropy and polarization of the CMB radiation. In fact a comprehensive analysis of various modifications of the photon dispersion relation was done already in [30], and the term $\kappa_3\theta^{\mu\nu}\varepsilon_\mu^{\rho\sigma\tau}F_{\rho\sigma}\square F_{\nu\tau}$ which we obtain here was included. Let us shortly discuss it. From the free-photon part of the effective action

$$\int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \kappa_3\theta^{\mu\nu}\varepsilon_\mu^{\rho\sigma\tau}F_{\rho\sigma}\square F_{\nu\tau} \quad (5.5)$$

we obtain the equation of motion

$$\partial^\alpha F_{\alpha\beta} - \kappa_3\theta^{\mu\nu}(2\varepsilon_{\mu\alpha\beta\sigma}\eta_{\rho\nu} + \varepsilon_{\mu\rho\sigma\beta}\eta_{\alpha\nu} - \varepsilon_{\mu\rho\sigma\alpha}\eta_{\beta\nu})\partial^\alpha\square F^{\rho\sigma} = 0. \quad (5.6)$$

Comparing (5.6) to equations and to notation of [31] we can identify

$$(k_F)_{\beta\alpha\rho\sigma} = \kappa_3\theta^{\mu\nu}(\varepsilon_{\mu\alpha\beta\sigma}\eta_{\nu\rho} - \varepsilon_{\mu\alpha\beta\rho}\eta_{\nu\sigma} + \varepsilon_{\mu\rho\sigma\beta}\eta_{\nu\alpha} - \varepsilon_{\mu\rho\sigma\alpha}\eta_{\nu\beta}). \quad (5.7)$$

It is easy to see however that due to identity (A.9), k_F vanishes! Therefore in fact the additional θ -linear term does not change the propagation of free photons: they satisfy the Maxwell equations, $\partial^\alpha F_{\alpha\beta} = 0$. There is no vacuum birefringence of photons, that is, none in linear order in θ . Were it present, the comparison with the observational data done in [31] would give that the scale of noncommutativity is of order 30 TeV², which is roughly in agreement with previously obtained constraints [32, 33, 34].

¹The discussion here is confined only to divergences obtained in perturbation theory; the question of chiral anomalies has to be treated additionally, see [28, 29].

²That is, $k_F \sim \Lambda_{\text{NC}}^{-2} \sim 10^{-9}\text{GeV}^{-2}$.

Spinors however behave differently. The modified free spinor action

$$\int i\bar{\varphi} (\bar{\sigma}_\rho \partial^\rho + i\kappa_4 \theta^{\mu\nu} \bar{\sigma}_\rho \sigma_{\mu\nu} \partial^\rho \square) \varphi + \text{h.c.}, \quad (5.8)$$

implies the equation

$$(i\bar{\sigma}^\rho \partial_\rho + i\kappa_4 \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \bar{\sigma}^\sigma \partial^\rho \square) \varphi = 0, \quad (5.9)$$

and we easily see that in this case the dispersion changes. Let us assume that noncommutativity is spatial, $\theta^{0i} = 0$, and denote $(\theta^{12})^2 = \theta_\perp^2$, $(\theta^{13})^2 + (\theta^{23})^2 = \theta_\parallel^2$; the momentum is along the third axis, $k^\mu = (E, 0, 0, p)$. The dispersion relation becomes

$$k^2 \left(1 - 4\kappa_4^2 \theta_\parallel^2 p^2 k^2 - 4\kappa_4^2 (\theta_\perp^2 + \theta_\parallel^2) k^4 \right) = 0 \quad (5.10)$$

and has the solutions

$$k^2 = 0, \quad k^2 = \frac{\sqrt{\frac{1}{\kappa_4^2} (\theta_\perp^2 + \theta_\parallel^2) + \theta_\parallel^4 p^4} - \theta_\parallel^2 p^2}{2(\theta_\perp^2 + \theta_\parallel^2)}. \quad (5.11)$$

One of the propagating fermionic modes acquires mass which is for small noncommutativity very large, of order $1/\sqrt{\theta}$, and thus on cosmological distances it is effectively suppressed. As the mass depends on the direction of propagation with respect to noncommutativity $\theta^{\mu\nu}$ this mode is birefringent.

The possibility of photon birefringence due to noncommutativity was first discussed in [5] within a classical θ -expanded gauge model. It was obtained that the effect exists in linear order only if there is an external electromagnetic field, otherwise the birefringence is of second order in θ . This result was expanded in [7]. Here also the first-order birefringence of photons exists in the external field but not in vacuum. Modifications of the photon propagator due to quantum corrections were thoroughly analyzed in many papers within the non-expanded noncommutative U(1) theory, [6, 8, 9]. However, as the theory is not perturbatively renormalizable it is not clear how to interpret the quantum corrections and to relate them to observations [35]. The analysis within a nonperturbative numerical approach was done in [10].

On the other hand, birefringence of chiral fermions obtained here is a completely new effect: it is absent for Dirac particles, [18, 26]. As astrophysical effects related to fermion propagation, for example for neutrinos, are very weak, it is not clear whether such effect can be tested experimentally in astrophysical measurements; perhaps high energy experiments would prove better for this task. In any case, physical implications of the obtained model need to be analyzed in more details and we plan to study them in our future work.

Therefore perhaps the main importance of the presented result is that another noncommutative gauge model with good renormalizability properties is found, and that it can be used as a building block for constructing noncommutative generalizations of the Standard Model. As we mentioned, a class of such models was found in [20, 21, 22]. In these papers, requirement of renormalizability (at one loop, in θ -linear order and on-shell) singled out GUT-compatible and anomaly-safe θ -expanded theories. Technically, these are the theories in which the left-handed and the conjugate of the right-handed fermion are in the same representation of the gauge group. Renormalizability implied that the triple gauge boson interactions were absent. Our model is somewhat different: it includes only one, say left-handed fermion; renormalizability is also one-loop and θ -linear but off-shell, and the triple gauge-boson interactions are

a priori allowed. Our framework is less restricting; however to achieve renormalizability we need the Seiberg-Witten redefinition of all fields.

But also in our model one can see that the GUT-compatibility is an important requirement. Let us assume that besides the left-handed spinor φ we also have a right-handed spinor $\bar{\chi}^\pm$ of the same or of the opposite charge³. Repeating the calculations for the one-loop correction of the gauge field propagator we obtain

$$\Gamma_{2,A} = \frac{1 \pm 1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\rho\sigma\tau} (\partial_\lambda F^{\rho\lambda}) (\partial_\nu F^{\sigma\tau}), \quad (5.12)$$

while the divergence of the fermion propagators is given by

$$\Gamma_{2,\varphi} = -\frac{1}{12} \frac{q^2}{(4\pi)^2 \epsilon} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \left(i(D^\rho \bar{\varphi}) \bar{\sigma}^\sigma (D^2 \varphi) \pm i(D^\rho \bar{\chi}^\pm) \bar{\sigma}^\sigma (D^2 \chi^\pm) + \text{h.c.} \right). \quad (5.13)$$

In both results of course both bosons and fermions run in the loop. The difference in signs in (5.12-5.13) comes from the fact that the action for the GUT-compatible spinor χ^- differs from the action for the χ^+ by the change $\theta \rightarrow -\theta$, [20]. Therefore if the model contains the pair (φ, χ^-) , the bosonic divergence vanishes. Analogously, it is easy to see that all gauge field redefinitions vanish too. This emphasises the fact that divergent term $\Gamma_{2,A}$ comes from, and depends on the fermion-boson interaction, and in specific cases the fermion loops cancel. For this reason also in the case of pure gauge U(1) and SU(N) theory there is no gauge field redefinition at linear order, [14, 15].

The present result opens new perspectives, while some of the old questions remain. The first and perhaps really nontrivial one is whether the field redefinitions are enough to ensure renormalizability also in quadratic order in θ . Though this question is technically very hard, it could happen that some additional Ward identities can help to resolve it, and we hope that this issue will be addressed and clarified in the future.

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A Conventions

We use the following chiral representation of γ -matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (A.1)$$

with

$$\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}) \quad (A.2)$$

and

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \quad \bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu). \quad (A.3)$$

³Notation χ^\pm is taken to be in accordance with [20].

In the field redefinitions we use

$$-i\varepsilon_{\mu\nu\rho\tau}\bar{\sigma}^\tau = \bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_\rho + \eta_{\mu\rho}\bar{\sigma}_\nu - \eta_{\nu\rho}\bar{\sigma}_\mu - \eta_{\mu\nu}\bar{\sigma}_\rho, \quad (\text{A.4})$$

$$\bar{\sigma}^\mu\sigma^{\nu\rho} + \bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu = -i\varepsilon^{\mu\nu\rho\tau}\bar{\sigma}_\tau \quad (\text{A.5})$$

Chiral spinors φ, χ multiply as

$$\varphi\chi = \chi\varphi, \quad \bar{\varphi}\bar{\chi} = \bar{\chi}\bar{\varphi}, \quad (\text{A.6})$$

$$\bar{\varphi}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\varphi}, \quad (\chi\sigma^\mu\bar{\varphi})^\dagger = \varphi\sigma^\mu\bar{\chi}. \quad (\text{A.7})$$

Those relations, as can be seen easily, give the usual identities for Majorana spinors ϕ, ψ

$$\begin{aligned} \bar{\phi}\psi &= \bar{\psi}\phi, \quad \bar{\phi}\gamma_5\psi = \bar{\psi}\gamma_5\phi, \\ \bar{\phi}\gamma^\mu\psi &= -\bar{\psi}\gamma^\mu\phi, \quad \bar{\phi}\gamma^\mu\gamma_5\psi = \bar{\psi}\gamma^\mu\gamma_5\phi. \end{aligned} \quad (\text{A.8})$$

Majorana lagrangians are obtained from the corresponding chiral ones using identities (A.6-A.7) and hermiticity of the lagrangian.

Finally relation between ε and η tensors (Schouten identity) reads

$$\varepsilon_{\mu\nu\rho\sigma}\eta_{\tau\lambda} + \varepsilon_{\mu\nu\tau\rho}\eta_{\sigma\lambda} - \varepsilon_{\mu\nu\tau\sigma}\eta_{\rho\lambda} + \varepsilon_{\tau\mu\rho\sigma}\eta_{\nu\lambda} - \varepsilon_{\tau\nu\rho\sigma}\eta_{\mu\lambda} = 0. \quad (\text{A.9})$$

Multiplying (A.9) by $\theta^{\mu\nu}F^{\rho\sigma}D^\lambda$ we obtain another useful relation

$$\theta^{\mu\nu}F^{\rho\sigma}(2\varepsilon_{\mu\rho\sigma\tau}D_\nu + 2\varepsilon_{\mu\nu\rho\tau}D_\sigma - \varepsilon_{\mu\nu\rho\sigma}D_\tau) = 0. \quad (\text{A.10})$$

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